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Kinetics of Bose-Condensation

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Abstract

The process of condensation in the system of scalar Bosons with weak $\lambda\phi^4$ interaction is considered. Boltzmann kinetic equation is solved numerically. Bose condensation proceeds in two stages: At the first stage condensate is still absent but there is non-zero inflow of particles towards $\vec{p} = 0$ and the distribution function at $\vec{p} = 0$ grows from finite values to infinity. At the second stage there are two components, condensate and particles, reaching their equilibrium values. We show that the evolution in both stages proceeds in a self - similar way and find the time needed for condensation, which is finite.

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It is a fundamental result of quantum statistics of Bosons that above a certain critical density all added particles must enter the ground state: Bose-Einstein condensate forms. The kinetics of this process is a very interesting problem. One can reach the Bose condensation gradually decreasing temperature in a sequence of equilibrium states. An appropriate description of this is given by the well-known kinetics of second order phase transitions. On the other hand, when the conditions for the formation of Bose condensate appear, the system can be far from the equilibrium. Recently this problem attracted particular interest in connection with exciting prospects for the experimental observation of Bose condensation in a very cold atomic samples, e.g. in a gas of spin-polarized atomic hydrogen [1] or in alkali-metal vapors [2]. Another interesting application of Bose kinetics is rather far from the laboratory experiments and is related to the problem of Bose stars formation [3,4] from the dark matter in the universe.

The question of the time evolution of weakly interacting Bose gas was addressed in a number of papers. In earlier treatments an ideal Bose gas was coupled to a thermal bath with infinite heat capacity [5,6]. Small energy exchange was assumed and after several other approximations a Fokker-Planck type equation was obtained. Levich and Yahkot [6] calculated analytically that the time for condensation is infinite in this situation. By including Boson-Boson interactions they later [7] found solution which describes explosive appearance of a condensate, but they concluded that this effect could have been an artifact of their approximations.

Snoke and Wolfe in Ref. [8] undertook direct numerical integration of Boltzmann kinetic equation. Although this calculation demonstrated the restructuring of the distribution function, the appearance of a Bose condensate was not detected. Their approach is close to ours but in comparison with the Ref. [8] we perform numerical integration in much wider dynamical range of relative energies and densities and we directly analyse the behavior the distribution function.

Another result recently reported in Ref. [9] states that the time required for condensation is $\sim T_c$, where T_c is the temperature of Bose condensation. This result seems to be incorrect

(or at least it can not be applied to all temporal stages of gas evolution) since it is insensitive to the interactions. Clearly, in the limit of zero couplings the distribution of particles does not evolve and the relaxation time has to be infinite.

Analytical study of Bose condensation was performed recently in the paper [10]. Three different regimes of evolution where identified. It was argued that in the kinetic region in non-linear regime the distribution function has to be a power law $f \propto \varepsilon^{-7/6}$. This power low is well known in the theory of plasma turbulence [11,12]. We indeed observe the tendency to this law in our numerical simulations, but the system never reaches it. Moreover, this distribution became destroyed with condensate appearance, contrary to the assumptions of Ref. [10]. What concerns condensation time, only rough dimensional estimations were done in Ref. [10].

We found that Bose condensation process can be divided on two stages. During the first part of the first stage the self-similar solution forms and then the distribution function reaches infinity at $\vec{p} = 0$ in a self-similar way. We can not observe in principle the actual build up of the coherence in the frameworks of Boltzmann equation. But we have to conclude that after the distribution function became infinite at zero momentum, the condensate had formed. While due to fluctuations the width of the coherent region can be finite in momentum space, we simply model the coherent field by $N_c\,\delta(\vec{\mathbf{p}})$ with initially infinitesimally small but then growing amplitude $N_c(t)$. We found that in this second stage the evolution also proceeds in a self-similar fashion and found the durations of both stages.

While our prime interest and motivations for this work was connected to the physics of Bose star, the problem of condensate formation with account for the gravitational field was never approached and we do not attempt it here. Instead, we are solving kinetic equations in flat space-time and only range of parameters of the initial distribution reflect the virial equilibrium of self-gravitating system.

We consider the system of scalar bosons with 4-particle self-interaction. The field-theoretical Lagrangian which we bear in mind is $L = (\partial_\mu\phi)^2/2 - m^2\phi^2/2 - \lambda\phi^4/4!$. Time development of the quantum state which contain well defined and large number of particles

can be adequately described by the Boltzmann kinetic equation which governs the evolution of one-particle distribution function, $f(\vec{\mathrm{p}})$:

$$\frac{df(\vec{\mathrm{p}}_1)}{dt} = \frac{\pi^4}{m^4} \int |M_{fi}|^2 F(f) \delta^4(\sum_i p_{\mu i}) \prod_{i=2}^{4} \frac{d^3\vec{\mathrm{p}}_i}{(2\pi)^3} \ , \tag{1}$$

where

$$F(f) = [1 + f_1]\,[1 + f_2] f_1' f_2' - [1 + f_1']\,[1 + f_2'] f_1 f_2 \ , \tag{2}$$

and $f_i \equiv f(\vec{p}_i)$, $f_i' \equiv f(\vec{p}_i{}')$. The equilibrium solution of the kinetic equation is the Bose-Einstein distribution function

$$f(\vec{\mathrm{p}}) = \frac{1}{\exp\left((\varepsilon - \mu)/T\right) - 1} + (2\pi)^3 N_c \delta(\vec{\mathrm{p}}) \ , \tag{3}$$

where ε is the particle energy, μ is the chemical potential, T is the temperature of the final state and N_c is the number density of particles in condensate.

In what follows we shall consider isotropic initial distribution $f = f(\varepsilon)$. In our case the matrix element is given by $|M_{fi}| = \lambda^2$ and the kinetic equation for the case without condensate, $N_c = 0$, can be rewritten in the form

$$\frac{df(\varepsilon_1)}{dt} = \frac{\lambda^2}{64\pi^3 m} \int\int F(f) \frac{D}{p_1} d\varepsilon_1' d\varepsilon_2' \equiv I_P \ , \tag{4}$$

where $D \equiv \min[p_1, p_2, p_1', p_2']$ and $\varepsilon_2 = \varepsilon_1' + \varepsilon_2' - \varepsilon_1$ in arguments of $F(f)$. The integration should be done over the region $\varepsilon_1 < \varepsilon_1' < \infty$, $\varepsilon_1 - \varepsilon_1' < \varepsilon_2' < \infty$.

After the moment of condensate formation the kinetic equation (4) is inappropriate for numerical integration anymore, the finite number of particles in condensate corresponds to the infinite value of the distribution function at zero energy. In order to describe the system of particles interacting with the condensate we divide the distribution function into two pieces: $\tilde{f} = f(\varepsilon, t) + (2\pi)^3 N_c(t) \delta^3(\vec{\mathrm{p}})$, where the first term corresponds to the "gas" of particles and the second one describe the condensate. Substituting this function into the original kinetic equation (1) we obtain

$$\dot{N}_c(t) = \frac{\lambda^2 N_c(t)}{64\pi^3 m} \int_0^\infty d\varepsilon_1' d\varepsilon_2' [f_1' f_2' - f_2(f_1' + f_2')] \ , \tag{5a}$$

$$\dot{f}(\varepsilon_1) = I_P + \frac{N_c(t)\lambda^2}{32\pi m^2 p_1} \left(\int_0^{\varepsilon_1} [(f_2' - f_1) f_1' - f_2' f_1] d\varepsilon_2' + 2 \int_{\varepsilon_1}^\infty [(f_2' - f_1) f_2 + f_2' f_1] d\varepsilon_2' \right) \ . \tag{5b}$$

In general, after the condensate formation (and at large particle densities even before) the description in terms of quasiparticles rather then particles is more appropriate. For example, the kinetic equation, Eq.(5), does not include the processes where one of the incoming and one of the outcoming particles has zero momentum, $p_2' = p_2 = 0$. This process does not contribute to the collision integral directly, i.e. it does not change the distribution of particles over energies, but it does change the effective particle mass.

However, in many cases those effects are insignificant and we still can work in terms of particles. The quantitative arguments are the following. The effective mass of quasiparticles in the presence of condensate (or in dense medium) is $m_{\rm eff} = m^2 + \lambda\phi_c^2/6 = m^2 + \lambda n/m$. We can still use the description in terms of particles if the second term in the sum is much smaller than the first one. Since $n \sim m^3(\Delta v)^3 f_0$ we obtain $\lambda f_0 (\Delta v)^3 \ll 1$, where Δv is characteristic velocity dispersion. In the case of axion miniclusters, for example, we have [4] $\lambda f_0 \sim 10^{-5}$; $\Delta v \sim 10^{-8}$, and the description in terms of particles is perfectly good.

As an initial distribution we choose the function $f(\varepsilon)$ which has the maximum at $\vec{\mathbf{p}} = 0$. In general, such distribution function can be characterized by means of three major parameters: (1) The overall amplitude f_0. In what follows we define $f_0 = f(\varepsilon = 0)$. (2) The energy scale ε_0 where the distribution function became twice smaller, $f(\varepsilon_0) = f_0/2$. (3) The effective width, Γ, of the region over which the distribution function varies rapidly. More specifically, we choose the initial distribution function to be of the form:

$$f(\varepsilon) = \frac{2f_0}{\pi} \arctan\left[\exp\left(\Gamma(1 - \varepsilon/\varepsilon_0)\right)\right] \ . \tag{6}$$

In what follows we shall measure the energy in units of ε_0 and the distribution function in units of f_0, i.e. initial distribution function will have the normalization $f(\varepsilon = 0) = 1$.

We define the dimensionless time τ as [3]:

$$\tau \equiv \frac{\varepsilon_0^2 f_0(1+f_0)\lambda^2}{64\pi^3 m}\, t\ . \tag{7}$$

The parameters ε_0 and f_0 in the limit $f_0 \gg 1$ after rescaling will not enter the kinetic equation explicitly, but will define the time scale. In terms of τ there remains the weak dependence of the relaxation time upon the initial shape parameter Γ. All data presented in this paper will correspond to one and the same value of $\Gamma = 5$. With the initial shape of the distribution function being given, the two scaling parameters f_0 and ε_0 define also the parameters of final equilibrium $\{T, \mu, N_c\}$. With $\Gamma = 5$ the choice $f_0 \leq f_{\rm crit} \approx 2.8$ corresponds to the Bose-gas without condensate, while $f_0 > f_{\rm crit}$ corresponds to condensate formation in the final equilibrium state.

We shall consider here the case $f_0 > 2.8$ only. We shall simplify the problem and consider $f_0 \gg 1$. In this case we can disregard f^2 terms in the function (2), which became $F(f) = [f_1 + f_2]f_1' f_2' - [f_1' + f_2']f_1 f_2$.

We integrated the kinetic equation in the energy interval $10^{-9} < \varepsilon < 10$. We defined the distribution function on the grid of 200 points equally spaced in the logarithm of energy and used the spline interpolation when calculating distribution function at intermediate points. For each integration in collision integral we had implied Gauss algorithm. Particle and energy non-conservation was of order 10^{-3} for the whole time of integration.

Results of numerical integration of the kinetic equation (4) are presented in Fig. 1, where we plot the distribution function at different moments of time. We have arranged the output each time when $f(\varepsilon = \varepsilon_{\rm min}, t)$ had increased by one order of magnitude. The most striking feature of this plot is self-similar character of the evolution. The distribution function has the "core" where $f(\varepsilon) \approx$ const and the radius of "core" decreases with time while the value of $f(\varepsilon)$ in the core grows. Outside the core the distribution function is the power law $f(\varepsilon) \propto \varepsilon^{-\alpha}$ and does not depend upon time to a very good accuracy. Self-similar solutions exhibiting this kind of behavior can be parametrized as

$$f(\varepsilon, \tau) = A^{-\alpha}(\tau) f_s(\varepsilon / A(\tau))\ , \tag{8}$$

where it is assumed $A(\tau) \to 0$ with the increase of time and we always can choose the normalization $f_s(0) = 1$. It is possible to find the time dependence of distribution function at $\varepsilon = 0$ at late times analytically, using self-similarity of the solution.

Substituting parametrization (8) in the kinetic equation we obtain:

$$f(0,\tau) = [2C(\tau_c - \tau)(\alpha - 1)]^{-\alpha/(2(\alpha-1))}\,, \tag{9}$$

and $f(0,\tau)$ has the pole at the finite moment of time $\tau = \tau_c$, $f(0,\tau_c) = \infty$.

The value of logarithmic derivative of $f(\varepsilon)$ is plotted in Fig. 1b. The dotted line in Figs. 1 corresponds to the limiting value $\alpha = 7/6$. This power law corresponds to the stationary solution of the kinetic equation (4), see Ref. [12]. But this value is never reached prior to the moment of condensate formation (after that moment the character of evolution completely changes). Rather, with the boundary condition $df/d\varepsilon = 0$ at $\varepsilon = 0$ the power law on the tail is $\alpha \approx 1.24$ (with appropriate boundary conditions our code correctly finds the root $\alpha = 7/6$). And the function Eq. (9) is a good fit to our data with $\alpha \approx 1.24$ and $\tau_c \approx 19$.

We conclude, that after the solution have reached self-similar form, the time dependence of the distribution function at zero momentum is given by $f(0,\tau) \propto (\tau_c - \tau)^{-2.6}$, which reaches infinity at finite time and means the onset of condensation.

Though the system does not reach the limiting value of $\alpha = 7/6$, at least this power law is the root of the equation $I_P = 0$ [12]. Because of that we selected the function $f(\varepsilon) \propto \varepsilon^{-7/6}$ at $\varepsilon < 1$ as an initial condition while integrating Eqs. (5). We took initially $N_c \ll N_{tot}$ (the particular choice for N_c, as far as this condition is satisfied, is insignificant).

Results of numerical integration of the system (5) are presented in Fig. 2. The dashed line corresponds to the initial distribution. Solid lines correspond to the distribution function at different moments of time. Basically, evolution proceeds in the following way. First, the power law $f(\varepsilon) \propto 1/(\varepsilon)^{7/6}$ changes to the law $f(\varepsilon) \propto 1/\varepsilon$ at small energies. And then this change propagates to the region of lager energies, see Fig. 2. Later on the power law stays on the equilibrium value $\alpha = 1$, but the amplitude of $f(\varepsilon)$ gradually decreases.

Again, we see that the essential parts of curves in Fig. 2 repeat itself under translation

from the left to the right and evolution is self-similar. During this epoch (before "the wave of change" had reached exponential tail of the initial distribution at $\varepsilon > 1$) approximately 40% of particles had condensed. And what is important, the number of particles in condensate linearly grows with time at this epoch, $N_c/N_{\rm tot} = B\tau$. This enables us to eliminate the ambiguity in the initial value for N_c since B does not depend upon it. Indeed, in our simulations which was done in a finite energy interval, during several first iterations system self-adjusts: a proper profile of distribution forms while condensate reaches particular value of N_c. We can disregard this period and extrapolate curves in Fig. 2, $N_c(\tau)$ and self-similar character of the evolution back in time and to the region of smaller energies.

We obtained $B \approx 2 \times f_i(1)^2$ (initial normalization $f_i(1) = 1$ corresponds roughly to the magnitude of final distribution at $\varepsilon \sim \varepsilon_{\rm min}$ in Fig. 1). For completeness we also had runs when we used final distribution in Fig. 1 as initial condition for integration Eqs. (5). Condensation in this case decelerates and N_c does not grow lineraly, all is in accordance with excess magnitude of the distribution function at small energies when $\alpha \approx 1.24$.

We have studied numerically the kinetics of Bose condensation of the weakly interacting Bose gas. The picture we observe differs even qualitatively from the previous works. The distribution function of excess particles which eventually has to form the condensate does not narrows in time gradually approaching δ- function, as it was found in Ref. [6] assuming small energy exchange per collision. Instead, power low profile $f \sim \varepsilon^{-7/6}$ tend to form which corresponds to the constant flux of particles in momentum space towards the condensate [12]. One could expect, as it was done in Ref. [10], that this very natural regime will persist in the presence of condensate as well, till all excess particles will inflow from the high energy tail and reside in the ground state. Nevertheless, this is not the case also and at the moment the condensate appears, it terminates this regime. Instead of this steady flow through the whole energy interval, particles from all energy levels jump directly to the condensate retaining during the major period of time the equilibrium shape of the distribution function, $f \sim \varepsilon^{-1}$. The constant of proportionality in this law gradually decreases till it reaches equilibrium value.

We can not, in principle, observe the build up of coherence being in frameworks of Boltzmann kinetic equation, but kinetic description has to be valid prior to and after the very moment of condensate formation. We have seen that evolution on both stages is self-similar. This allows us to obtain a number of useful analytical relations, e.g. the time dependence of the distribution function near the point of condensate formation, Eq. 9. We have found the duration of both stages, which is finite and of order $\tau_c \sim 20$.

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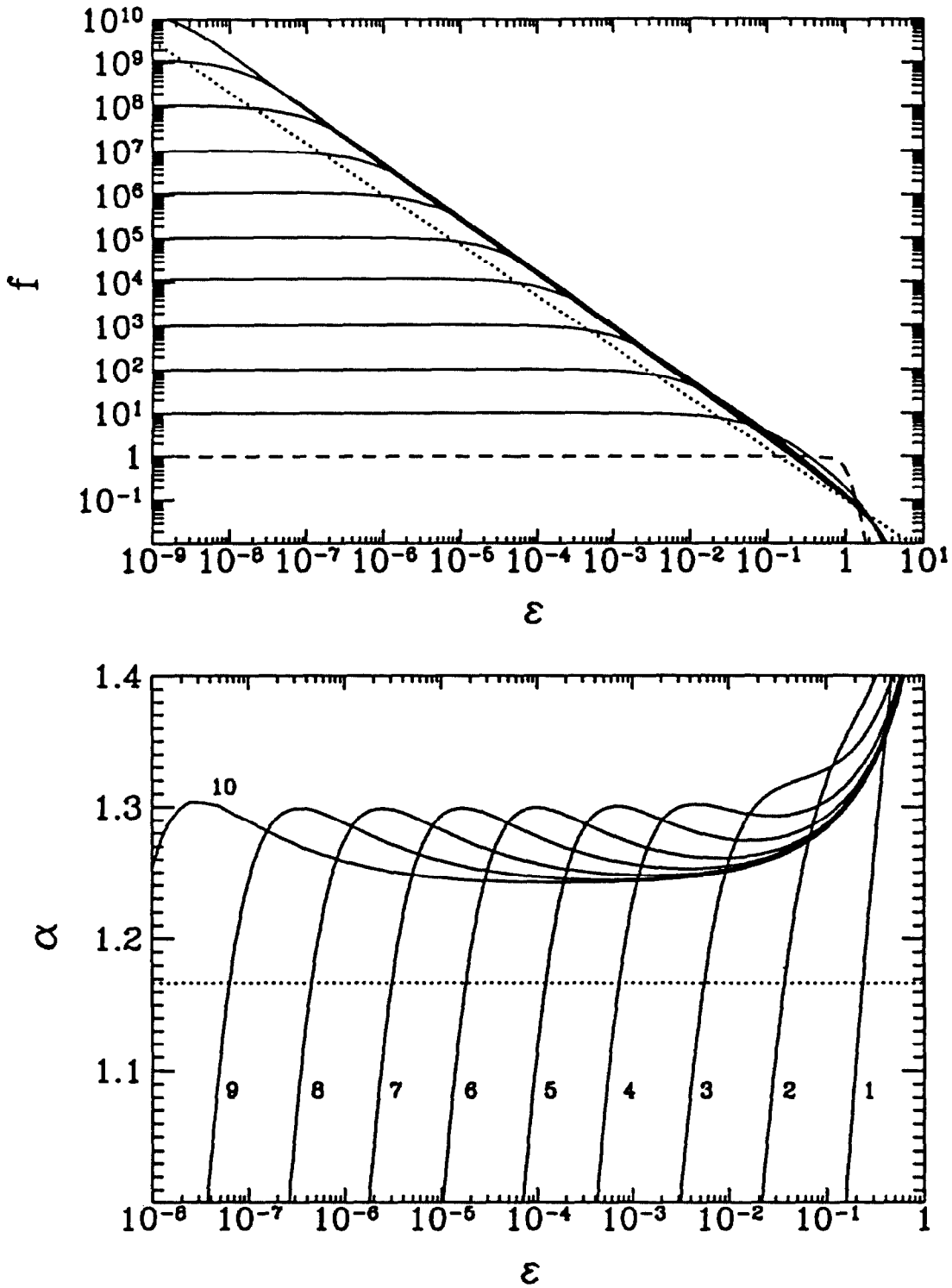

FIG. 1. The distribution function (top) and its logarithmic derivative (bottom) are shown at different moments of time prior to condensate formation.

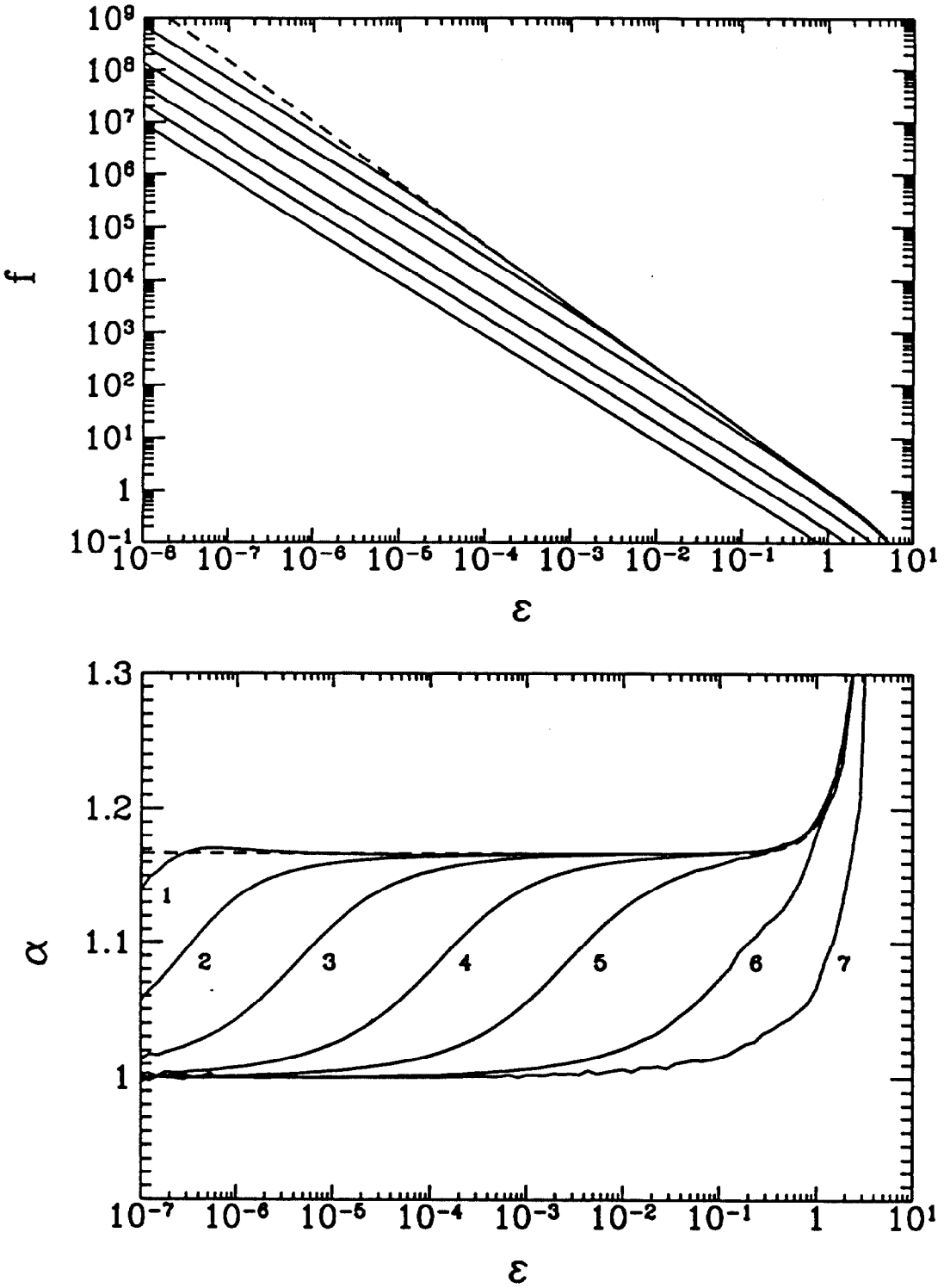

FIG. 2. The distribution function (top) and its logarithmic derivative (bottom) are shown at different moments of time during condensation